Complex numbers: what the course uses

Amplitudes are complex numbers, so this is the mathematical ground the course stands on: two notations, four operations, and the one rule that powers interference.

1 · Two ways to write the same number

$$c = a + b\,i \ \ (\text{cartesian}) \qquad c = r\,e^{i\varphi} \ \ (\text{polar}) \qquad e^{i\varphi} = \cos\varphi + i\sin\varphi$$

i is the number with $i^2 = -1$. Draw c as a point in the plane: real part a horizontal, imaginary part b vertical. The polar view describes the same point by its distance from the origin r, the **magnitude**, and its angle φ, the **phase**, measured in radians. Euler's formula is the bridge between the two notations.

$$r = |c| = \sqrt{a^2 + b^2} \qquad \varphi = \operatorname{atan2}(b, a) \qquad a = r\cos\varphi \qquad b = r\sin\varphi$$

2 · The operations

Addition is componentwise, cartesian wins: (a+bi) + (c+di) = (a+c) + (b+d)i. **Multiplication** in cartesian form expands the brackets and uses $i^2 = -1$; in polar form it is where polar wins: multiply the magnitudes, add the phases.

$$(a+bi)(c+di) = (ac - bd) + (ad + bc)\,i, \qquad r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2\, e^{i(\varphi_1 + \varphi_2)}$$

The **conjugate** flips the sign of the imaginary part and gives the magnitude squared without square roots. **Division** multiplies by the conjugate of the denominator to make it real.

$$\bar{c} = a - b\,i = r\,e^{-i\varphi}, \qquad c\,\bar{c} = |c|^2 = a^2 + b^2, \qquad \frac{1}{c} = \frac{\bar{c}}{|c|^2}$$

3 · Worked examples

Question	Working
$(1 + i)^2$	$1 + 2i + i^2 = 1 + 2i - 1 = 2i$
1 / i	multiply top and bottom by i: $i / i^2 = i / (-1) = -i$
$\lvert 3 + 4i\rvert^2$	$(3 + 4i)(3 - 4i) = 9 + 16 = 25$, so $\lvert 3 + 4i\rvert = 5$; note $(3 + 4i)^2 = -7 + 24i$ is something else
e^(iπ/2) · e^(iπ/2)	add the phases: e^(iπ) = cos π + i sin π = −1
2e^(iπ/4) · 3e^(iπ/4)	magnitudes 2 · 3 = 6, phases π/4 + π/4 = π/2: 6e^(iπ/2) = 6i
|e^(iφ)| for any real φ	$\cos^2\varphi + \sin^2\varphi = 1$, so the magnitude is always 1

4 · Radians and the angles that keep coming back

Every rotation gate takes its angle in radians, and NumPy's trigonometric functions expect radians. A full turn is 2π, a right angle π/2. Feeding degrees to np.cos gives wrong numbers without any error message.

φ (radians)	0	π/4	π/2	π	3π/2	2π
degrees	0°	45°	90°	180°	270°	360°
cos φ	1	0.707	0	−1	0	1
sin φ	0	0.707	1	0	−1	0
e^(iφ)	1	(1+i)/√2	i	−1	−i	1

5 · The rule the course lives on

Multiplying by $e^{i\varphi}$, a number of magnitude exactly one, changes no magnitude and only turns the phase. That is a **phase shift**: rotation without scaling, invisible to a measurement on its own, and the entire mechanism of interference once paths recombine (Session 2). The Born rule turns an amplitude c into a probability $|c|^2 = c \cdot \bar{c}$, which is why c^2 is never a probability: squaring keeps the phase, the magnitude-square discards it.

In NumPy

Quantity	Call
Build one	c = 2 + 3j — j is Python's i; np.array([…], dtype=complex)
Magnitude · phase	np.abs(c) · np.angle(c) (radians)
Real · imaginary part	c.real · c.imag
Conjugate	np.conj(c) — or c.conjugate()
Polar to cartesian	r * np.exp(1j * phi)
Probability from an amplitude	np.abs(c)**2 — never c**2
Degrees to radians	np.radians(45) — or 45 * np.pi / 180

Refresher, if any of this felt rusty

About an hour. Or paste the question you missed into an LLM and ask for the idea behind it, then redo the worked examples above by hand.

Resource	By	Time	Link
Imaginary Numbers Are Real, part 1 (continue with parts 2 to 5)	Welch Labs	40 min	youtu.be/T647CGsuOVU
$e^{i\pi}$ in 3.14 minutes, using dynamics	3Blue1Brown	4 min	youtu.be/v0YEaeIClKY
Cheat sheet: vectors and matrices (the companion sheet)	qubit-lab.ch	3 pages	qubit-lab.ch/Quantum_Kickstart_Cheat_Sheet_Vectors_Matrices.pdf

> **Two pitfalls.** $|c|^2 = a^2 + b^2$, not $(a + b)^2$. And c^2 is not $|c|^2$: the first keeps the phase, the second discards it, which is exactly why measurement cannot see phase.